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ON THE STRESS CONCENTRATION IN INCLUSIONS IN COMPOSITE MATERIALS

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Elastic composite media consisting of a homogeneous and isotropic matrix in which the other components are distributed in the form of ellipsoidal inclusions are considered. The location and orientation of the inclusions in the matrix are random, therefore, a random stress field (microstresses) is generated in the composite under strain. The microstresses reach a maximum on the surface of the inclusions (stress concentration effect). In this paper we establish a dependence between these maximum microstresses and the mean microstresses for a moderate concentration of inclusions. The known results [1] of an investigation of the state of stress in the neighborhood of an ellipsoidal inclusion in an unbounded elastic medium are elucidated in Sect. 1. The required stress field satisfies an integral equation whose solution can be found exactly [2]. The problem of determining the microstress field in a composite material reduces to solving a system of integral equations (Sect.2). The large number of equations as well as the randomness of the location of the centers of the inclusions in the matrix, which exclude the possibility of an exact solution of this system, permit the use of the statistical nature of the problem and its approximate solution. Consequently, the effective tensor stress concentration coefficient, relating the configurational means over the surface of the isolated inclusion to the macrostresses, is determined. This quantity is represented in the form of the convolution of two tensors, one of which is the stress concentration coefficient on an isolated inclusion in the matrix, while the other takes account of the influence of the residual inclusions. A correcting tensor is calculated in Sect. 3 for certain kinds of macroscopic isotropic composites and, particularly for a medium weakened by circular cracks. A circular crack in a composite with spherical inclusions is considered in Sect. 4 and the effective stress intensity coefficient is found as a function of the stiffness and the volume content of the inclusions.

1. Let us consider an ellipsoidal inhomogeneity occupying a domain v in an unbounded medium with the elastic moduli tensor L_0. At great distances from the inclusion let stresses or displacements be given. The strain field $\varepsilon(\mathbf{x})$ in such a medium satisfies the integral equation [1]

$$\varepsilon(\mathbf{x}) = \varepsilon_0(\mathbf{x}) + \int_v G(\mathbf{x}-\mathbf{x}')[L]\,\varepsilon(\mathbf{x}')\,d\mathbf{x}' \tag{1.1}$$

$$G(\mathbf{x}) = (G_{ijkl}(\mathbf{x})) = [U_{ik,\,jl}(\mathbf{x})]_{(ij)(kl)}, \quad [L] = L_1 - L_0$$

Here $\varepsilon_0(\mathbf{x})$ is the strain field which would exist in a medium without inclusions, L_1 is the elastic moduli tensor of the inclusion, $U_{ik}(\mathbf{x})$ is the Green's tensor of the Lamé

equations for the fundamental medium, parentheses denote symmetrization with respect to the corresponding subscripts, and the product of two tensors is understood to be the convolution relative to the two subscripts: $Ab \equiv A_{ijkl}b_{kl}$, $AB \equiv A_{ijmn}B_{mnkl}$.
An equation analogous to (1.1) can also be written for the stresses

$$\sigma(\mathbf{x}) = \sigma_0(\mathbf{x}) + \int_v \Gamma(\mathbf{x}-\mathbf{x}')[M]\,\sigma(\mathbf{x}')\,d\mathbf{x}' \qquad (1.2)$$

$$\sigma_0(\mathbf{x}) = L_0\varepsilon_0(\mathbf{x}),\ \Gamma(\mathbf{x}) = -L_0(I\delta(\mathbf{x}) + G(\mathbf{x})L_0)$$

$$[M] = M_1 - M_0,\ M_1 = L_1^{-1},\ M_0 = L_0^{-1},\ I = (I_{ijkl}) = \delta_{i(k}\delta_{l)}$$

where I_{ijkl} is a unit quadrivalent tensor, and $\delta(\mathbf{x})$ is the delta-function. If $\mathbf{x} \in v$, then (1.2) is converted into an integral equation relative to the stress field $\sigma^+(\mathbf{x})$ in the inclusions. In particular, the field σ^+ for a homogeneous field σ_0 is also homogeneous and determined by the expression

$$\sigma^+ = B\sigma_0,\ B = (I + Q[M])^{-1}$$

$$Q = L_0(I - PL_0),\quad P = -\int_v G(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}',\quad (\mathbf{x},\mathbf{x}' \in v) \qquad (1.3)$$

where P is a constant quadrivalent tensor dependent on the geometric characteristics of the ellipsoid and on the elastic moduli of the basic material.

If the matrix is isotropic, then

$$P_{ijkl} = \frac{1}{\mu_0}\left(\delta_{ik}\varphi^+_{,jl} - \frac{3k_0+\mu_0}{3k_0+4\mu_0}\psi^+_{,ijkl}\right)_{(ij)(kl)} \qquad (1.4)$$

where k_0, μ_0 are the volume and shear elastic moduli of the medium φ^+, ψ^+ are the harmonic and biharmonic potentials of an ellipsoid of unit density at an interior point. It hence follows that the tensor P must have the symmetry of the ellipsoid and be determined by nine real components. In a coordinate system coincident with the principal axes of the ellipsoid

$$P_{1111} = \gamma_0[3J_{11} + (1-4\nu_0)J_1],\ P_{1122} = \gamma_0(J_{21} - J_1) \qquad (1.5)$$
$$P_{1212} = \gamma_0[J_{21} + J_{12} + (1-2\nu_0)(J_1 + J_2)],\ \gamma_0 = [16\pi\mu_0(1-\nu_0)]^{-1}$$

(ν_0 is the Poisson's ratio). The quantities

$$J_p = \frac{3}{2}v\int_0^\infty \frac{du}{(a_p^2+u)\Delta(u)},\quad J_{pq} = \frac{3}{2}va_p^2\int_0^\infty \frac{du}{(a_p^2+u)(a_q^2+u)\Delta(u)}$$
$$(\Delta = [(a_1^2+u)(a_2^2+u)(a_3^2+u)]^{1/2},\ v = {}^4/_3\,\pi a_1a_2a_3)$$

where a_i $(i = 1, 2, 3)$ are the lengths of the ellipsoid semi-axes which are expressed in terms of elliptic integrals. The remaining six nonzero components of the tensor P_{ijkl} are obtained from (1.5) by a circular commutation of the subscripts.

Outside the inclusion the stress field is

$$\sigma^-(\mathbf{x}) = \sigma_0 - L_0G^-(\mathbf{x})L_0[M]\sigma^+ \qquad (1.6)$$

$$G^{-}(\mathbf{x}) = \int_{v} G(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}', \quad (x \overline{\in} v)$$

Substituting (1.3) into (1.6) yields

$$\sigma^{-}(\mathbf{x}) = F_0(\mathbf{x})\,\sigma_0, \quad F_0(\mathbf{x}) = B\,\{I + (Q - L_0 G(\mathbf{x})\, L_0)\,[M]\} \tag{1.7}$$

This expression permits determination of the limit value of the stress tensor $\sigma^{-}(\mathbf{n})$ on the inclusion boundary from outside. There is hence no need to calculate the external ellipsoid potentials. Indeed, by using the relation for jumps in the derivatives of the potentials upon passing through the boundary

$$[\varphi_{,jl}] = -4\pi n_j n_l, \quad [\psi_{,ijkl}] = -8\pi n_i n_j n_k n_l$$

where n_i are components of the unit normal to the surface of the inclusion, we obtain

$$\sigma^{-}(\mathbf{n}) = B L_0 M_1 \{I + K(\mathbf{n})\,[L]\}\,\sigma_0 \tag{1.8}$$

$$K(\mathbf{n}) = (K_{ijkl}(\mathbf{n})) = \frac{1}{\mu_0}\left[\delta_{ik} n_j n_l - \frac{3k_0 + \mu_0}{3k_0 + 4\mu_0}\, n_i n_j n_k n_l\right]_{(ij)(kl)}$$

Finally, taking into account that B can be converted as follows:

$$B = M_0 L_1 A, \quad A = (I + P\,[L])^{-1}$$

we arrive at an expression for the tensorial stress concentration coefficient $F_0(\mathbf{n})$

$$\sigma^{-}(\mathbf{n}) = F_0(\mathbf{n})\,\sigma_0, \quad F_0(\mathbf{n}) = A\,\{I + K(\mathbf{n})\,[L]\} \tag{1.9}$$

which agrees with that found in [1] for the more general case by using the problem of the connection of two media.

2. Now, let us examine a multicomponent composite material consisting of a homogeneous matrix and filler particles of ellipsoidal shape. For simplicity, we shall consider the inclusions of one component of identical magnitude but differently oriented in space.

Let us isolate the characteristic volume V of the composite, i.e., a volume with dimensions substantially exceeding the distance between the inclusions, but within whose limits the change in the macroscopic stress and strain fields can be neglected. Such a volume should contain a sufficient number of inclusions for averaging, where the material within its limits can be considered macroscopically homogeneous.

Let us represent the tensor of elastic compliances $M(\mathbf{x})$ as

$$M(\mathbf{x}) = M_0 + \delta M(\mathbf{x}), \quad \delta M(\mathbf{x}) = \sum_{\alpha=1}^{n} [M_\alpha] \sum_{m=1}^{N_\alpha} v_{\alpha m}(\mathbf{x}) \tag{2.1}$$

$$[M_\alpha] = M_\alpha - M_0$$

where n is the number of components distributed in the material in the form of inclusions, N_α is the number of inclusions of the α - th component in the characteristic volume, and $v_{\alpha m}(\mathbf{x})$ is a function equal to one within the m - th inclusion of the α - th component and zero outside the inclusion. The stress field $\sigma(\mathbf{x})$ in V satisfies an equation analogous to (1.2)

$$\sigma(\mathbf{x}) = \sigma_0 + \int_{V} \Gamma(\mathbf{x}, \mathbf{x}')\,\delta M(\mathbf{x}')\,\sigma(\mathbf{x}')\, d\mathbf{x}' \tag{2.2}$$

$$\Gamma(\mathbf{x}, \mathbf{x}') = -L_0\,[I\delta(\mathbf{x}-\mathbf{x}') + G(\mathbf{x}, \mathbf{x}')\,L_0]$$

Here G is defined exactly as in Sect. 1, where $U(x, x')$ is the Green's tensor of the Lame equation for a medium with elastic moduli L_0 which vanishes on the surface S of the volume V, and σ_0 is the external field which would exist in the matrix for given boundary conditions on S. These latter are selected in such a way that the field σ_0 would be homogeneous. We shall henceforth consider the volume V so large that the Green's tensor $U(\mathbf{x}, \mathbf{x}')$ in the expression for the kernel in (2.2) could be replaced by the Green's tensor $U(\mathbf{x} - \mathbf{x}')$ for an unbounded domain.

According to (2.1), the field of the tensor $\delta M(\mathbf{x})$, and hence, the form under the integral sign in (2.2) are not zero only in subdomains occupied by the inclusions. Therefore, this integral reduces to the sum of integrals in such subdomains and (2.2) becomes

$$\sigma(\mathbf{x}) = \sigma_0 + \sum_\alpha \sum_m \int_{v_{\alpha m}} \Gamma(\mathbf{x} - \mathbf{x}')[M_\alpha]\, \sigma(x')\, d\mathbf{x}' \qquad (2.3)$$

where $v_{\alpha m}$ is the volume of the m - th inclusion of the a - th component. If the characteristic volume contains

$$N = \sum_\alpha N_\alpha$$

inclusions, then taking the arbitrary point $\mathbf{x} \in v_{\alpha m}$ as the point $\mathbf{x}$ for $m = 1, 2, \ldots$, we obtain a system of linear singular integral equations in the N tensor functions $\sigma(\mathbf{x})$ $(\mathbf{x} \in v_{\alpha m})$ which describe the stress fields in the inclusions. An exact solution of this system is quite a complex problem even for relatively small N. On the other hand, namely a large number and a random location of the centers of inclusions in the characteristic volume permit using the statistical nature of the problem and finding the mathematical expectation of the quantities required. In particular, the stress field can be determined in an arbitrary inclusion in V, averaged over that set of inclusion configurations in the matrix for which the position and orientation of this inclusion have been fixed. The tensorial stress concentration factor in the inclusion, determined on the basis of this solution, should be understood as the mean in precisely that sense.

Let us fix the point $\mathbf{x}$ in an arbitrary k – th inclusion of the s - th component and let us rewrite (2.3) as follows

$$\sigma(\mathbf{x}) = \tau(\mathbf{x}) + \int_{v_{sk}} \Gamma(\mathbf{x} - \mathbf{x}')[M_s]\, \sigma(\mathbf{x}')\, d\mathbf{x}', \quad \mathbf{x}, \mathbf{x}' \in v_{sk} \qquad (2.4)$$

Here integration over the domain v_{sk} has been extracted and the following notation has been introduced:

$$\tau(\mathbf{x}) = \sigma_0 + \sum_{\alpha, m}' \int_{v_{\alpha m}} \Gamma(\mathbf{x} - \mathbf{x}_{\alpha m} - \xi)[M_\alpha]\, \sigma(\mathbf{x}_{\alpha m} + \xi)\, d\xi \qquad (2.5)$$

In this expression $\mathbf{x}_{\alpha m}$ is the radius - vector of the center of the inclusion $v_{\alpha m}$ and ξ is a vector connecting the center of the inclusion to its arbitrary inner point and the prime on the summation sign indicates the absence of a term with subscript $m = k$ for $\alpha = s$.

Now consider the ensemble of inclusions in the volume V. For each realization of the ensemble $\sigma = \sigma(\mathbf{x}; \mathbf{F}_N)$, where F_N is a set of radius - vectors of the centers of the inclusions and their orientations governing the specific configuration. We introduce the conditional distribution function $\varphi(\mathbf{x}_{sk}, \boldsymbol{\omega}_{sk} \mid \mathbf{F}_{N-1})$, where $\boldsymbol{\omega}_{sk}$ is the set of Euler angles giving the orientation of an ellipsoid with center at $\mathbf{x}_{sk}$. Applying the operation of taking the average over this distribution function to (2.4), we obtain

$$\langle \sigma(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle = \langle \tau(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle + \int_{v_{sk}} \Gamma(\mathbf{x} - \mathbf{x}')[M_s] \langle \sigma(\mathbf{x}' \mid \mathbf{x}_{sk}) \rangle \, d\mathbf{x}' \qquad (2.6)$$

Let us use (2.5) and the identity

$$\varphi(\mathbf{x}, \omega \mid \mathbf{F}_{N-1}) = \varphi(\mathbf{x}, \omega \mid \mathbf{x}', \omega') \, \varphi(\mathbf{x}, \omega; \mathbf{x}', \omega' \mid \mathbf{F}_{N-2}) \qquad (2.7)$$

where $\varphi(\mathbf{x}, \omega \mid \mathbf{x}', \omega')$ is the conditional unary distribution function normalized to unity, to evaluate the quantity $\langle \tau(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle$. For a statistically homogeneous distribution of the centers of the inclusion with a moderate concentration we set

$$\varphi(\mathbf{x}, \omega \mid \mathbf{x}', \omega') = H(|\mathbf{x} - \mathbf{x}'| - 2a) \, \varphi(\mathbf{x}, \omega), \quad \varphi(\mathbf{x}, \omega) = \frac{1}{V} \varphi(\omega) \qquad (2.8)$$

where $H(\mathbf{x})$ is the Heaviside function and a is the major semi-axis of the ellipsoid. Taking the average of both sides of (2.5) with (2.7) and (2.8) taken into account, we obtain

$$\langle \tau(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle = \sigma_0 + \frac{1}{V} \sum_{\alpha, m}{}' \iiint \Gamma(\mathbf{x} - \mathbf{x}_{\alpha m} - \xi)[M_\alpha] \times \qquad (2.9)$$
$$H(|\mathbf{x} - \mathbf{x}_{\alpha m}| - 2a) \, \varphi(\omega_{\alpha m}) \langle \sigma(\xi \mid \mathbf{x}_{\alpha m}; \mathbf{x}_{sk}) \rangle \, d\xi \, d\omega_{\alpha m} \, d\mathbf{x}_{\alpha m}$$

Here $\langle \sigma(\xi \mid \mathbf{x}_{\alpha m}; \mathbf{x}_{sk}) \rangle$ is the field $\sigma(\mathbf{x})$ $(\mathbf{x} \in v_{\alpha m})$ averaged over the set of configurations for which the position and orientation of two inclusions are fixed. Let us assume that $\langle \sigma(\xi \mid \mathbf{x}_{\alpha m}; \mathbf{x}_{sk}) \rangle = \langle \sigma(\xi \mid \mathbf{x}_{\alpha m}) \rangle$, where because of the homogeneity of the field σ_0 this quantity is independent of the location of the inclusion in the volume V but depends only on the orientation $\omega_{\alpha m}$. Then using regularizations of the integrals which diverge at infinity, by virtue of which [3]

$$\int G(\mathbf{x} - \mathbf{x}') \, d\mathbf{x}' = M_0$$

we reduce (2.9) to the form

$$\langle \tau(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle = \sigma_0 - Q_0 \sum_\alpha [M_\alpha] \, c_\alpha \langle \bar{\sigma}_\alpha \rangle_\omega \qquad (2.10)$$

$$Q_0 = -L_0(I - P_0 L_0), \quad \langle f \rangle_\omega \equiv \int f(\omega) \, \varphi(\omega) \, d\omega,$$

$$c_\alpha = \frac{N_\alpha}{V} \sum_{m=1}^{N_\alpha} v_{\alpha m}$$

Here $\bar{\sigma}_\alpha(\omega)$ is the stress field in an arbitrary inclusion of the α-th component averaged over the volume of the inclusion, and the components of the isotropic tensor P_0 are determined by (1.5) in which we should put $J_p = {}^4/_3\,\pi$, $J_{pp} = 3\,J_{pq} = {}^4/_5\,\pi a^{-2}$ $(a_i = a)$. Therefore, under the assumptions made, the field $\langle \tau(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle$ turns out to be homogeneous. Then as follows from (2.6), the field $\langle \sigma(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle$ is also homogeneous and it can be identified with the mean relative to the inclusion. Using the notation $\langle \sigma(\mathbf{x} \mid \mathbf{x}_{sk}) \rangle = \sigma_s(\omega_k)$, we have in conformity with (1.3)

$$\sigma_s(\omega_k) = B_s(\omega_k) \Big[\sigma_0 - Q_0 \sum_\alpha [M_\alpha] \, c_\alpha \langle \bar{\sigma}_\alpha \rangle_\omega \Big] \qquad (2.11)$$

$$B_s(\omega_k) = (I + Q(\omega_k)[M_s])^{-1}, \quad Q(\omega_k) = L_0(I - P(\omega_k) L_0)$$

Multiplying both sides of (2.11) by $c_s [M_s]$, taking the average with respect to the orientations and summing over all components, we find

$$\sum_\alpha [M_\alpha] c_\alpha \langle\sigma_\alpha\rangle_\omega = D \sum_\alpha [M_\alpha] c_\alpha \langle B_\alpha\rangle_\omega \sigma_0 \tag{2.12}$$

$$D = \left(I - Q_0 \sum_\alpha [M_\alpha] c_\alpha \langle B_\alpha\rangle_\omega\right)^{-1} \tag{2.13}$$

Substituting (2.12) into the right side of (2.11) results in the following final expression for the stress tensor $\sigma_s^+(\omega)$ within the ellipsoid of the s-th phase with the orientation ω:

$$\sigma_s^+(\omega) = B_s(\omega) D \langle\sigma\rangle \tag{2.14}$$

It has hence been taken into account that σ_0 coincides with the macrostresses $\langle\sigma\rangle$ in conformity with the definition of the characteristic volume of the composite.

If the point of observation $\mathbf{x}$ in the matrix is fixed near the boundary of the isolated inclusion, then by using reasoning analogous to that presented above, we obtain the expression

$$\sigma_s^-(\mathbf{x}) = D F_{0s}(\mathbf{x}) \langle\sigma\rangle \tag{2.15}$$

for the stress field $\sigma_s^-(\mathbf{x})$ in the neighborhood of this inclusion, where $F_{0s}(\mathbf{x})$ is defined by (1.7) as before. It hence follows that the tensorial stress concentration factor in the inclusions in composite materials

$$F_s(\mathbf{n}) = D F_{0s}(\mathbf{n}) \tag{2.16}$$

differs from the concentration factor $F_{0s}(\mathbf{n})$ in a single inclusion in the matrix by the tensorial factor $D = (D_{ijkl})$ which takes account of the influence of the other inclusions.

3. The stress concentration in an ellipsoidal inhomogeneity in an elastic medium and, particularly, in an ellipsoidal crack and needle has been investigated in detail in [1,4]. Hence, without examining $F_{0s}(\mathbf{n})$ further we present the value of the correction tensor D for certain kinds of composite materials.

Let the inclusions in the composite material be isotropic with the volume and shear elastic moduli k_α and μ_α, and let their orientations be equally probable. We denote the volume concentrations of the inclusions of the different components by $c_\alpha (\sum_\alpha c_\alpha + c_0 = 1$, where c_0 is the relative volume of the matrix). In this case the tensor $\langle B_\alpha\rangle_\omega$ is isotropic, i.e.,

$$\langle B_{ijkl}^\alpha\rangle = B_1^\alpha \delta_{ij}\delta_{kl} + 2 B_2^\alpha (I_{ijkl} - {}^1/_3\, \delta_{ij}\delta_{kl}) \tag{3.1}$$

$$B_1{}^\alpha = {}^1/_9 B_{iijj}^\alpha, \quad B_2{}^\alpha = {}^1/_{10}(B_{ijij}^\alpha - 3B_1{}^\alpha) \tag{3.2}$$

The tensor D is also isotropic with the components

$$D_{ijkl} = D_1 \delta_{ij}\delta_{kl} + D_2 (I_{ijkl} - {}^1/_3\, \delta_{ij}\delta_{kl}) \tag{3.3}$$

$$D_1 = \left[1 - 3q_1^\circ \sum_\alpha c_\alpha \left(\frac{1}{k_\alpha} - \frac{1}{k_0}\right) B_1{}^\alpha\right]^{-1}$$

$$D_2 = \left[1 - 2q_2^\circ \sum_\alpha c_\alpha \left(\frac{1}{\mu_\alpha} - \frac{1}{\mu_0}\right) B_2^\alpha\right]^{-1} \tag{3.4}$$

$$q_1^\circ = \frac{4\mu_0 k_0}{3k_0 + 4\mu_0}, \quad q_2^\circ = \frac{\mu_0 (9k_0 + 8\mu_0)}{5(3k_0 + 4\mu_0)}$$

Let us assume that the composite contains spherical inclusions of only one component. In this case $P = P_0$ and (3.4) simplifies

$$D_1 = \frac{1}{3}\left[1 + q_1^\circ\left(\frac{1}{k_1} - \frac{1}{k_0}\right)\right]\left[1 + c_0 q_1^\circ\left(\frac{1}{k_1} - \frac{1}{k_0}\right)\right]^{-1} \tag{3.5}$$

$$D_2 = \left[1 + q_2^\circ\left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right)\right]\left[1 + c_0 q_2^\circ\left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right)\right]^{-1}$$

In particular, if the stiffness of the inclusions substantially exceeds the stiffness of the matrix, then

$$D_1 = \frac{k_0 - q_1^\circ}{3(k_0 - c_0 q_1^\circ)}, \quad D_2 = \frac{\mu_0 - q_2^\circ}{\mu_0 - c_0 q_2^\circ}$$

i.e., the stress concentration on the surface of the inclusions diminishes with the growth in their volume content. In the other limiting case when the matrix contains spherical vacancies (pores), the tensor D takes an especially simple form

$$D_{ijkl} = c_0^{-1} I_{ijkl}$$

and, therefore, the stresses on the pore surface grow with the increase in porosity.

Let us consider a porous material whose pores have the shape of ellipsoids of revolution with the semi-axes $a_1 = a_2 = a > a_3$ and the ratio between the semi-axes $\eta = a_3/a$. In this case

$$D = (I - c_1 Q_0 \langle A \rangle_\omega)^{-1} \tag{3.6}$$

where the tensor A^{-1} has an orthorhombic structure with six nonzero real components

$$A_{1111}^{-1} = A_{2222}^{-1} = \varkappa_0 [1 - {}^1/_8 (3 - {}^3/_4 f_1 + f_2)] \tag{3.7}$$

$$A_{1122}^{-1} = \varkappa^0 \{\nu_0 - {}^1/_{16} [2 - {}^1/_2 f_1 - 2(1 - 4\nu_0) f_2]\}$$

$$A_{1133}^{-1} = A_{2233}^{-1} = \varkappa_0 \{\nu_0 - {}^1/_{16} [(1 + \eta^2) f_1 - (1 - 4\nu_0) \times (4 - f_2)]\}$$

$$A_{3333}^{-1} = {}^1/_4 \varkappa_0 (\eta^2 f_1 + f_2)$$

$$A_{1212}^{-1} = \varkappa_0 \left\{\frac{1 - \nu_0}{2} - \frac{1}{16}\left[2 - \frac{1}{2} f_1 + 2(1 - 2\nu_0) f_2\right]\right\}$$

$$A_{1313}^{-1} = A_{2323}^{-1} = \varkappa_0 \left\{\frac{1 - \nu_0}{2} - \frac{1}{16} [(1 + \eta^2) f_1 + (1 - 2\nu_0)(4 - f_2)]\right\}$$

$$f_1 = \frac{4 - 3f_2}{1 - \eta^2}, \quad f_2 = \frac{2\eta}{(1 - \eta^2)^{3/2}} (\arccos \eta - \eta\sqrt{1 - \eta^2})$$

$$\varkappa_0 = \frac{2\mu_0}{1 - \nu_0}$$

in a coordinate system with x_3 - axis coincident with the axis of rotation of the ellipsoid.

The formulas presented permit the investigation of the stress concentration on the surfaces of circular cracks in planform, which are distributed in the material (we understand the crack to be an ellipsoidal cavity with the ratio $\eta = a_3/a$ tending to zero). In this case, a difficulty occurs in evaluating the matrix $A^{-1}(\eta)$ which is associated with the fact that it becomes singular, i.e., $\det A^{-1}(\eta \to 0) \to 0$. To realize the passage to the limit as $\eta \to 0$, corresponding to the crack, we write the expansion of the tensor $A^{-1}(\eta)$ in a series in η

$$A^{-1}(\eta) = A_0^{-1} + A_1^{-1}\eta + O(\eta^2)$$

where the first two terms presented for the expansion are obtained from (3.7) for $f_1 = 4 - 3\eta$, $f_2 = \eta$. Then we invert the tensor $A_0^{-1} + A_1^{-1}\eta$ which takes account of only terms of the order of $1/\eta$. We consequently obtain

$$A_{3333} = \frac{4(1-\nu_0^2)}{\pi(1-2\nu_0)\eta}, \quad A_{1313} = A_{2323} = \frac{2(1-\nu_0)}{\pi(2-\nu_0)\eta}$$

and the remaining components are zero to $O(1)$ accuracy. Substituting these expressions in to (3.6), we find that the tensor D for an equally probable crack orientation is determined by (3.3) in which

$$D_1 = (1 - 3cq_1^\circ A_1)^{-1}, \quad D_2 = (1 - 2cq_2^\circ A_2)^{-1}$$

$$A_1 = \frac{4}{9\pi}\frac{1-\nu_0^2}{1-2\nu_0}, \quad A_2 = \frac{4}{15\pi}\frac{(1-\nu_0)(5-\nu_0)}{2-\nu_0}, \quad c = \frac{4}{3}\pi n a^3$$

($n = N/V$ is the countable crack concentration)

4. Now, let us assume that a microcrack of radius a occurred in a composite material with spherical inclusions. We consider the material to have three components, where the role of one is played by a cavity in the form of an ellipsoid of revolution, which passes into a crack in the limit. To investigate the state of stress in the neighborhood of the crack, (2.15) can be used, in which the tensor D is determined by (3.3) and (3.4). However, the quantities D_1 and D_2 can be found by means of (3.5) because of the smallness of the factor a^3/V.

Let the macroscopic loading of the material reduce to simple tension along the x_1 axis, and the plane of the crack be perpendicular to this axis. In this case the stress σ_{11}^- at the crack edge has a singularity, and the effective stress intensity factor k^* is

$$k^* = (D_1 + {}^2/_3 D_2)\, k$$

where k is the stress intensity factor for a crack under tension in a medium without inclusions. If the inclusions are vacancies, then

$$k^* = k/c_0$$

and for absolutely rigid inclusions

$$k^* = \frac{1}{3}\left[\frac{k_0 - q_1^\circ}{k_0 - c_0 q_1^\circ} + \frac{2(\mu_0 - q_2^\circ)}{\mu_0 - c_0 q_2^\circ}\right] k$$

It is seen from the formulas presented that the stress intensity factor at the crack vertex will grow with the increase in porosity, while the presence of rigid inclusions diminishes k^* and therefore, increases the critical load.

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